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Some Results on Harmonic Mean Cordial Graphs

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ABSTRACT
All the graphs considered in this article are simple and undirected. Let $G = (V(G), E(G))$ be a simple undirected Graph. A function $f: V(G) \to \{1,2\}$ is called Harmonic Mean Cordial if the induced function $f^*: E(G) \to \{1,2\}$ defined by $f^*(uv) = \lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \rfloor$ satisfies the condition $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{1,2\}$, where $v_f(x)$ and $e_f(x)$ denotes the number of vertices and number of edges with label x respectively. A Graph G is called Harmonic Mean Cordial graph if it admits Harmonic Mean Cordial labeling. In this article, we have discussed Some Results on Harmonic Mean Cordial Graphs.

KEYWORDS
Harmonic Mean Cordial Labeling, Degree Splitting Graph, Path, Cycle, Shell, Gear

1. Introduction
The notion of graph labeling in graph theory has garnered significant attention from scholars because of its wide-ranging and rigorous applications in domains such as communication network design and analysis, military surveillance, social sciences, optimization, and linear algebra. Various graph labelings are documented in the current body of literature. A dynamic survey of graph labeling by Gallian is a condensed compilation of a lengthy bibliography of articles on the subject.

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$. For terminology and notation not defined here we have followed Balakrishnan and Rangnathan . In J. Gowri and J. Jayapriya defined Harmonic Mean Cordial labeling of graph G. Let $G = (V(G), E(G))$ be a simple undirected Graph. A function $f: V(G) \to \{1,2\}$ is called *Harmonic Mean Cordial* if the induced function $f^*: E(G) \to \{1,2\}$ defined by $f^*(uv) = \lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \rfloor$ satisfies the condition $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{1,2\}$, where $v_f(x)$ and $e_f(x)$ denotes the number of vertices and number of edges with label x respectively and $\lfloor x \rfloor$ is the floor function. A Graph G is called *Harmonic Mean Cordial graph* if it admits Harmonic Mean Cordial labeling. For the sake of convenience of the reader we use HMC for Harmonic Mean Cordial labeling. It is useful to recall some useful definitions of graph theory to make this article self−contained.

Motivated by the interesting results proved in and on Root Cube Mean Cordial Labeling in .

Definition 1. Let $G = (V(G), E(G))$ be a graph with $V(G) = V_1 \cup V_2 \cup \ldots \cup V_k \cup W$, where each V_i is a set of all the vertices having same degree with at least two elements and $W = V(G) \setminus \bigcup_{i=1}^{k} V_i$. The degree splitting graph $DS(G)$ is obtained from G by adding vertices $u_1, u_2, \ldots, u_t$ and joining to each vertex of V_i for $1 \leq i \leq k$.

Definition 2. A walk is called a path if all the vertices appearing in it are distinct. It is denoted by P_n.

Definition 3. Cycle is a closed trail in which all the vertices are distinct. It is denoted by C_n.
Definition 4. A Shell graph is defined as a cycle C_n with $(n-3)$ chords sharing a common end point called the apex. It is denoted by $C_{(n,n-3)}$.

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Definition 5. The gear graph G_n is obtained from wheel W_n by subdividing each of its rim edge.

Definition 6. Bistar $B_{n,n}$ is the graph obtained by joining the center(apex) vertices of two copies of $K_{1,n}$ by an edge.

Definition 7. The square of a graph G denoted by G^2 has the same vertex set as that of G and two vertices are adjacent in G^2 if they are at a distance of 1 or 2 apart in G.

Definition 8. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The Cartesian product of G_1 and G_2 which is denoted by $G_1 \times G_2$ is the graph with vertex set $V = V_1 \times V_2$ consisting of vertices $u = (u_1, u_2), v = (v_1, v_2)$ such that u and v are adjacent in $G_1 \times G_2$ whenever ($u_1 = v_1$ and u_2 adjacent to v_2) or ($u_2 = v_2$ and u_1 adjacent to v_1).

Definition 9. The Möbius ladder M_n is a graph obtained from the ladder $P_n \times P_2$ by joining the opposite end vertices of two copies of P_n.

Definition 10. Let P_n be a path on n vertices denoted by $(1,1), (1,2), \dots, (1, n)$ and with $n - 1$ edges denoted by $e_1, e_2, \dots, e_{n-1}$ where e_i is the edge joining the vertices $(1, i)$ and $(1, i + 1)$. On each edge $e_i, i = 1,2, \dots, n - 1$ we erect a ladder with $n - (i - 1)$ steps including the edge e_i. The graph so obtained is called a step ladder graph which is denoted by $S(T_n)$, where n denotes the number of vertices in the base.

Definition 11. The circular ladder graph CL_n is defined as $C_n \times P_2$.

In the next section, in Theorems 2.1 and 2.2, we investigate $DS(P_n)$ and $DS(S_n)$ for odd n are harmonic mean cordial graphs, respectively. Also, in Theorems 2.6, we have derived HMC labeling for square of path P_n^2 for odd n. In Theorem 2.3, 2.4, 2.5, 2.7, 2.8, 2.9, 2.10 and 2.11, we investigate $DS(C_n)$, $DS(S_n)$ for even n, $DS(G_n)$, $DS(B_{n,n})$, P_n^2 for even n, C_n^2, M_n, $S(T_n)$ and CL_n are not harmonic mean cordial graphs, respectively.

2. Main Results

Theorem 2.1. The degree splitting path $DS(P_n)$ is HMC for $n > 2$.
Proof. Let $G = (V, E) = DS(P_n)$ be the degree splitting path. Note that $|V| = n + 2$ and $|E| = 2n - 1$. Let $V = \{u, v, v_1, v_2, \dots, v_n\}$ be the vertex set of degree splitting path $DS(P_n)$ shown in the following figure - 1.
Case 1: n is odd
Define a labeling function $f: V(DS(P_n)) \to \{1,2\}$ as follows,
$f(u) = 2$,
$f(v) = 1$,
$f(v_1) = 1$,
$f(v_i) = 2$, if $2 \le i \le \left\lceil \frac{n}{2} \right\rceil + 1$,
$f(v_i) = 1$, if $\left\lceil \frac{n}{2} \right\rceil + 2 \le i \le n$
Then $v_f(1) = \frac{n+1}{2}, v_f(2) = \frac{n-1}{2}$ and $e_f(1) = n, e_f(2) = n - 1$. So, we have $|v_f(1) - v_f(2)| = 0$ and $|e_f(1) - e_f(2)| = 1$.
Case 2: n is even
Define a labeling function $f: V(DS(P_n)) \to \{1,2\}$ as follows,

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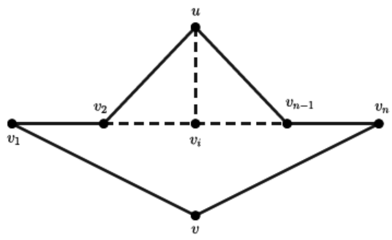
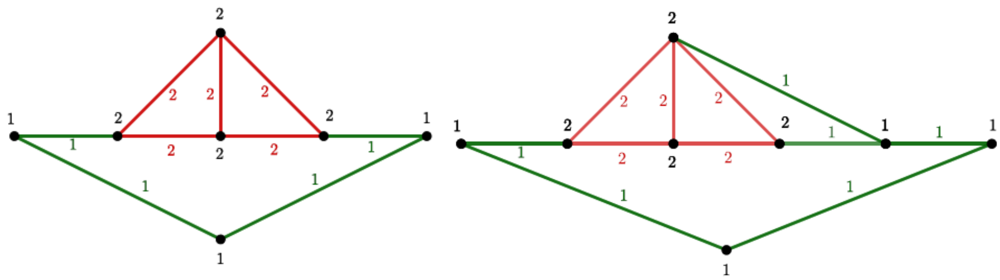


Figure – 1 : $DS(P_n)$

$f(u) = 2,$
$f(v) = 1,$
$f(v_1) = 1,$
$f(v_i) = 2, \text{if } 2 \leq i \leq \frac{n}{2} + 1$
$f(v_i) = 1, \text{if } \frac{n}{2} + 2 \leq i \leq n$
Then $v_f(1) = v_f(2) = n + 1$ and $e_f(1) = n, e_f(2) = n - 1$. So, we have $|v_f(1) - v_f(2)| = 0$ and $|e_f(1) - e_f(2)| = 1$.
Hence, The degree splitting path $DS(P_n)$ is HMC for $n > 2$.

Example 2.1. HMC labeling of $DS(P_5)$ and $DS(P_6)$ is shown in following figure - 2.



$DS(P_5)$ $DS(P_6)$

Figure 2

Theorem 2.2. The degree splitting Shell $DS(S_n)$ is HMC for odd n.
Proof. Let $G = (V, E) = DS(P_n)$ be the degree splitting Shell. Note that $|V| = n + 2$ and $|E| = 3n - 4$. Let $V = \{u, v, v_1, v_2, \ldots, v_n\}$ be the vertex set of degree splitting Shell $DS(S_n)$ shown in the following figure-3.

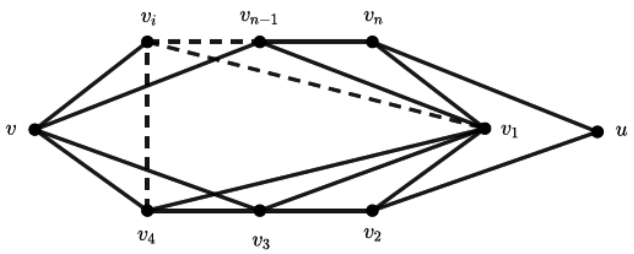


Figure – 3 : $DS(S_n)$

Define a labeling function $f: V(DS(S_n)) \to \{1,2\}$ as follows,
$f(u) = 1,$
$f(v) = 2,$
$f(v_1) = 2,$

$f(v_2) = 1,$
$f(v_i) = 2, \text{ if } 3 \le i \le \frac{n+3}{2}$
$f(v_i) = 1, \text{ if } \frac{n+5}{2} \le i \le n$
Then $v_f(1) = \frac{n+1}{2}, v_f(2) = \frac{n+3}{2}$ and $e_f(1) = \frac{3n-3}{2}, e_f(2) = \frac{3n-5}{2}$. So, we have $|v_f(1) - v_f(2)| = 1$ and $|e_f(1) - e_f(2)| = 1$.
Hence, The degree splitting Shell $DS(S_n)$ is HMC for odd n.

Example 2.3. HMC labeling of $DS(S_7)$ is shown in following figure - 4.

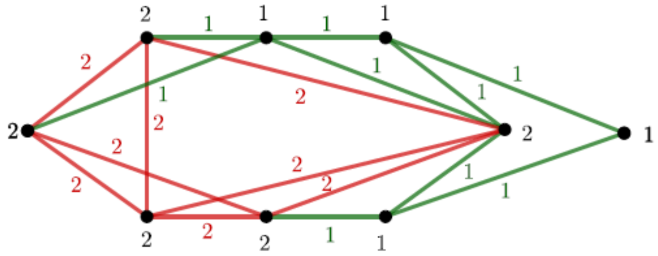


Figure – 4 : $DS(S_7)$

Theorem 2.3. The degree splitting Shell $DS(S_n)$ is not HMC for even n.
Proof. Let $G = (V, E) = DS(S_n)$ be the degree splitting Shell. Note that $|V| = n + 2$ and $|E| = 3n - 4$. Let $V = \{u, v, v_1, v_2, \dots, v_n\}$ be the vertex set of degree splitting Shell $DS(S_n)$.
Case 1: $n = 4$
If possible, let there be a HMC labeling $f: V(DS(S_4)) \to \{1,2\}$ for degree splitting Shell $DS(S_4)$.
The degree splitting Shell $DS(S_4)$ under consideration violates the condition as shown in below Table - 1.

n	$\lvert V(G)\rvert$	$\lvert E(G)\rvert$	$\lvert v_f(1)\rvert$	$\lvert v_f(2)\rvert$	$\lvert e_f(1)\rvert$	$\lvert e_f(2)\rvert$	$\lvert e_f(1) - e_f(2)\rvert$
4	6	9	3	3	6	3	3

Table - 1

Case 2: $n > 4$
If possible, let there be a HMC labeling $f: V(DS(S_n)) \to \{1,2\}$ for degree splitting Shell $DS(S_n)$.
So, $v_f(1) = v_f(2) = \frac{n+2}{2}$. If we assign consecutive labeling 2 on $\frac{n+2}{2}$ vertices of $DS(S_n)$, then we have $e_f(2) = \frac{3n-8}{2}$ and $e_f(1) = \frac{3n}{2}$. Therefore $|e_f(1) - e_f(2)| = 4 > 1$.
Without assuming consecutive labeling 2 on $\frac{n+2}{2}$ vertices of $DS(S_n)$, we get $|e_f(1) - e_f(2)| > 4$.
Hence, The degree splitting Shell $DS(S_n)$ is not HMC for even n.

Theorem 2.4. The degree splitting Gear $DS(G_n)$ is not HMC for all n.
Proof. Let $G = (V, E) = DS(G_n)$ be the degree splitting Gear. Note that $|V| = 2n + 3$ and $|E| = 5n$. Let $V = \{u, v, w, v_1, v_2, \dots, v_n, v_{n+1}, \dots, v_{2n}\}$ be the vertex set of degree splitting Gear $DS(G_n)$.
If possible, let there be a HMC labeling $f: V(DS(G_n)) \to \{1,2\}$ for degree splitting Gear $DS(G_n)$.
If we assign consecutive labeling 2 on $n + 2$ vertices of $DS(G_n)$, then we have $e_f(2) = 2n$ and $e_f(1) = \frac{3n}{2}$.
Therefore $|e_f(1) - e_f(2)| = n > 1$ as $n \ge 3$. Without assuming consecutive labeling 2 on $n + 2$ vertices of $DS(G_n)$, we get $|e_f(1) - e_f(2)| > n$ as $n \ge 3$.
Hence, The degree splitting Shell $DS(G_n)$ is not HMC for all n.

Theorem 2.5. The degree splitting Bistar $DS(B_{n,n})$ is not HMC for all n.
Proof. Let $G = (V, E) = DS(B_{n,n})$ be the degree splitting Bistar. Note that $|V| = 2n + 4$ and $|E| = 4n + 3$.

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Let $V = \{t, u, v, w, v_1, v_2, \dots, v_n, v_{n+1}, \dots, v_{2n}\}$ be the vertex set of degree splitting Bistar $DS(B_{n,n})$. If possible, let there be a HMC labeling $f: V\left(DS(B_{n,n})\right) \to \{1,2\}$ for degree splitting Bistar $DS(B_{n,n})$. If we assign consecutive labeling 2 on $n+2$ vertices of $DS(B_{n,n})$, then we have $e_f(2) = 2n$ and $e_f(1) = 2n+3$. Therefore $|e_f(1) - e_f(2)| = 3 > 1$ as $n \geq 3$.
Without assuming consecutive labeling 2 on $n+2$ vertices of $DS(B_{n,n})$, we get $|e_f(1) - e_f(2)| > 3$ as $n \geq 3$. Hence, The degree splitting Shell $DS(B_{n,n})$ is not HMC for all n.

Theorem 2.6. The square of path P_n^2 is HMC for odd n.
Proof. Let $G = (V, E) = P_n^2$ be the square of cycle. Note that $|V| = n$ and $|E| = 2n - 3$. Let $V = \{v_1, v_2, \dots, v_n\}$ be the vertex set of be the vertex set of square of path P_n^2 shown in the following figure - 5.

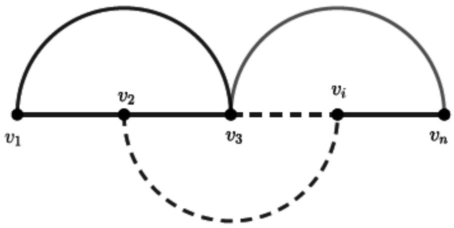

Figure – 5 : P_n^2

Define a labeling function $f: V(P_n^2) \to \{1,2\}$ as follows,

$$f(v_i) = \begin{cases} 2, & \text{if } 1 \leq i \leq \frac{n+1}{2} \\ 1, & \text{if } \frac{n+3}{2} \leq i \leq n \end{cases}$$

Then $v_f(1) = \frac{n-1}{2}, v_f(2) = \frac{n+1}{2}$ and $e_f(1) = n-1, e_f(2) = n-2$. So, we have $|v_f(1) - v_f(2)| = 1$ and $|e_f(1) - e_f(2)| = 1$.
Hence, The square of path P_n^2 is HMC for odd n.

Example 3. HMC labeling of P_7^2 is shown in following figure - 6.

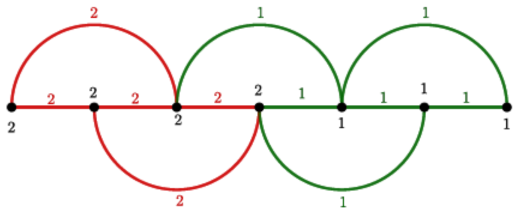

Figure – 6 : P_7^2

Theorem 2.7. The square of path P_n^2 is not HMC for even n.
Proof. Let $G = (V, E) = P_n^2$ be the square of cycle. Note that $|V| = n$ and $|E| = 2n - 3$. Let $V = \{v_1, v_2, \dots, v_n\}$ be the vertex set of square of cycle P_n^2.
If possible, let there be a HMC labeling $f: V(C_n^2) \to \{1,2\}$ for square of path P_n^2 graph.
So, $v_f(1) = v_f(2) = \frac{n}{2}$. If we assign consecutive labeling 2 on $\frac{n}{2}$ vertices of P_n^2, then we have $e_f(2) = n - 3$ and $e_f(1) = n$. Therefore $|e_f(1) - e_f(2)| = 3 > 1$.
Without assuming consecutive labeling 2 on $\frac{n}{2}$ vertices of P_n^2, we get $|e_f(1) - e_f(2)| > 3$.
Hence, The square of path C_n^2 is not HMC even n.

Theorem 2.8. The square of cycle C_n^2 is not HMC for all n.

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Proof. Let $G = (V, E) = C_n^2$ be the square of cycle. Note that $|V| = n + 1$ and $|E| = 2n$. Let $V = \{v, v_1, v_2, \dots, v_n\}$ be the vertex set of square of cycle C_n^2.
If possible, let there be a HMC labeling $f: V(C_n^2) \to \{1,2\}$ for square of cycle C_n^2 graph.
Case 1: n **is even**
So, $v_f(1) = v_f(2) = \frac{n}{2}$. If we assign consecutive labeling 2 on $\frac{n+1}{2}$ vertices of C_n^2, then we have $e_f(2) = n - 3$ and $e_f(1) = n + 3$. Therefore $|e_f(1) - e_f(2)| = 6 > 1$.
Without assuming consecutive labeling 2 on $\frac{n}{2}$ vertices of C_n^2, we get $|e_f(1) - e_f(2)| > 6$.
Case 2: n **is odd**
So, $v_f(1) = \frac{n-1}{2}, v_f(2) = \frac{n+1}{2}$. If we assign consecutive labeling 2 on $\frac{n+1}{2}$ vertices of C_n^2, then we have $e_f(2) = n - 2$ and $e_f(1) = n + 2$. Therefore $|e_f(1) - e_f(2)| = 4 > 1$.
Without assuming consecutive labeling 2 on $\frac{n}{2}$ vertices of C_n^2, we get $|e_f(1) - e_f(2)| > 4$.
Hence, The square of cycle C_n^2 is not HMC for all n.

Theorem 2.9. The Möbius Ladder M_n is not HMC for all $n \geq 2$.
Proof. Let $G = (V, E) = M_n$ be the Möbius Ladder. Note that $|V| = 2n$ and $|E| = 3n$. Let $V = \{v_1, v_2, \dots, v_n, v_{n+1}, \dots, v_{2n}\}$ be the vertex set of Möbius Ladder M_n.
If possible, let there be a HMC labeling $f: V(M_n) \to \{1,2\}$ for Möbius Ladder M_n graph.
Case 1: $n = 3$
If possible, let there be a HMC labeling $f: V(M_n) \to \{1,2\}$ for Möbius Ladder M_n.
The Möbius Ladder M_n under consideration violates the condition as shown in Table - 2.

n	$\lvert V(G)\rvert$	$\lvert E(G)\rvert$	$\lvert v_f(1)\rvert$	$\lvert v_f(2)\rvert$	$\lvert e_f(1)\rvert$	$\lvert e_f(2)\rvert$	$\lvert e_f(1) - e_f(2)\rvert$
3	6	9	3	3	7	2	5

Table - 2

Case 2: $n \neq 3$
If possible, let there be a HMC labeling $f: V(M_n) \to \{1,2\}$ for Möbius Ladder M_n.
So, $v_f(1) = v_f(2) = n$. If we assign consecutive labeling 2 on n vertices of M_n, then we have $e_f(2) = n$ and $e_f(1) = 2n$. Therefore $|e_f(1) - e_f(2)| = n > 1$.
Without assuming consecutive labeling 2 on n vertices of M_n, we get $|e_f(1) - e_f(2)| > n$.
Hence, The Möbius Ladder M_n is not HMC for all $n \geq 2$.

Theorem 2.10. The step ladder $S(T_n)$ is not HMC for all n.
Proof. Let $G = (V, E) = S(T_n)$ be the step ladder. Note that $|V| = \frac{n^2+5n+2}{2}$ and $|E| = n^2 + 3n$. Let $V = \{v_1, v_2, \dots, v_{\frac{n^2+5n+2}{2}}\}$ be the vertex set of step ladder $S(T_n)$.
If possible, let there be a HMC labeling $f: V(S(T_n)) \to \{1,2\}$ for step ladder $S(T_n)$.
Case 1: $\frac{n^2+5n+2}{2}$ **is even**
So, $v_f(1) = v_f(2) = \frac{n^2+5n+2}{4}$. If we assign consecutive labeling 2 on $\frac{n^2+5n+2}{4}$ vertices of $S(T_n)$, then we have $e_f(2) = \frac{n^2+3n-4}{2}$ and $e_f(1) = \frac{n^2+3n+4}{2}$. Therefore $|e_f(1) - e_f(2)| = 4 > 1$.
Without assuming consecutive labeling 2 on $\frac{n^2+5n+2}{4}$ vertices of $S(T_n)$, we get $|e_f(1) - e_f(2)| > 4$.
Case 2: $\frac{n^2+5n+2}{2}$ **is odd**
So, $v_f(1) = \lfloor\frac{n^2+5n+2}{4}\rfloor, v_f(2) = \lceil\frac{n^2+5n+2}{4}\rceil$. If we assign consecutive labeling 2 on $\lceil\frac{n^2+5n+2}{4}\rceil$ vertices of $S(T_n)$, then we have $e_f(2) = \frac{n^2+3n-4}{2}$ and $e_f(1) = \frac{n^2+3n+4}{2}$. Therefore $|e_f(1) - e_f(2)| = 4 > 1$.
Without assuming consecutive labeling 2 on $\frac{n^2+5n+2}{4}$ vertices of $S(T_n)$, we get $|e_f(1) - e_f(2)| > 4$.
Hence, The step ladder $S(T_n)$ is not HMC for all n.

Theorem 2.11. The Circular Ladder CL_n is not HMC for all $n \geq 2$.
Proof. Let $G = (V, E) = CL_n$ be the Circular Ladder. Note that $|V| = 2n$ and $|E| = 3n$. Let $V = \{v_1, v_2, \ldots, v_n, v_{n+1}, \ldots, v_{2n}\}$ be the vertex set of Circular Ladder CL_n.
If possible, let there be a HMC labeling $f: V(CL_n) \rightarrow \{1,2\}$ for Circular Ladder CL_n graph.
If possible, let there be a HMC labeling $f: V(CL_n) \rightarrow \{1,2\}$ for Circular Ladder CL_n.
So, $v_f(1) = v_f(2) = n$. If we assign consecutive labeling 2 on n vertices of CL_n, then we have $e_f(2) = n$ and $e_f(1) = 2n$. Therefore $|e_f(1) - e_f(2)| = n > 1$.
Without assuming consecutive labeling 2 on n vertices of CL_n, we get $|e_f(1) - e_f(2)| > n$.
Hence, The Circular Ladder CL_n is not HMC for all $n \geq 2$.

3. Conclusion

In this article, we have discussed $DS(P_n)$, $DS(S_n)$ for odd n and square of path P_n^2 for odd n are harmonic mean cordial graphs, respectively. Also, we investigate $DS(S_n)$ for even n, $DS(G_n)$, $DS(B_{n,n})$, P_n^2 for even n, C_n^2, M_n, $S(T_n)$ and CL_n are not harmonic mean cordial graphs, respectively.

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