

Harmonic Mean Cordial Labeling of Join of Some Graphs

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ABSTRACT

All the graphs considered in this article are simple and undirected. Let $G = (V(G), E(G))$ be a simple undirected Graph. A function $f : V(G) \rightarrow \{1,2\}$ is called Harmonic Mean Cordial if the induced function $f^* : E(G) \rightarrow \{1,2\}$ defined by $f^*(uv) = \left\lfloor \frac{(2f(u)f(v))}{(f(u)+f(v))} \right\rfloor$ satisfies the condition $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{1,2\}$, where $v_f(x)$ and $e_f(x)$ denotes the number of vertices and number of edges with label x respectively and $\lfloor x \rfloor$ denotes the greatest integer less than or equals to x . A Graph G is called Harmonic Mean Cordial graph if it admits Harmonic Mean Cordial labeling. In this article, we have discussed Harmonic Mean Cordial labeling of $C_n \vee W_m$ and $K_n \vee W_m$.

KEYWORDS

Harmonic mean cordial; Complete; Cycle; Wheel; Join of two graphs

1. Introduction

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$. For basic terminology and notation not defined here we have followed Balakrishnan and Ranganathan [1]. In [5] J. Gowri and J. Jayapriya defined Harmonic Mean Cordial labeling of graph G . Let $G = (V(G), E(G))$ be a simple undirected Graph. A function $f : V(G) \rightarrow \{1,2\}$ is called Harmonic Mean Cordial if the induced function $f^* : E(G) \rightarrow \{1,2\}$ defined by $f^*(uv) = \left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \right\rfloor$ satisfies the condition $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{1,2\}$, where $v_f(x)$ and $e_f(x)$ denotes the number of vertices and number of edges with label x respectively and $\lfloor x \rfloor$ is the floor function. A Graph G is called Harmonic Mean Cordial graph if it admits Harmonic Mean Cordial labeling. For the sake of convenience of the reader we use HMC for harmonic mean cordial labeling. Motivated by the interesting results proved on HMC labeling in [4, 5, 6, 7, 10, 11, 13] and on interesting results proved on Root Cube Mean Cordial labeling by [9, 12] in this article we have discussed HMC labeling of the Join graphs $C_n \vee W_m$ for $n \geq 3, m \geq 4$ and $K_n \vee W_m$ for $n \geq 2, m \geq 4$ and we have proved that they are not HMC. Here, we have mentioned basic definitions to make this article self-contained.

Definition 1. [1] A simple graph G is said to be complete if every pair of distinct vertices of x are adjacent in G . It is denoted by K_n .

Definition 2. [1] A walk in a graph G is a finite alternating sequence of vertices and edges. A walk is called a trail if all the edges are distinct. Cycle is a closed trail in which all the vertices are distinct. It is denoted by C_n .

Definition 3. [1] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then union of G_1 and G_2 is denoted by $G_1 \cup G_2$ is the graphs whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2$. When G_1 and G_2 are vertex disjoint $G_1 \cup G_2$ is called sum of G_1 and G_2 and it is denoted by $G_1 + G_2$.

Definition 4. [1] Let G_1 and G_2 be two vertex-disjoint graphs. Then the join of two graphs G_1 and G_2 denoted as $G_1 \vee G_2$ is a super graph of $G_1 + G_2$ in which every vertex of G_1 is adjacent to each vertex of G_2 .

Definition 5. [1] If $G_1 = K_1$ and $G_2 = C_n$; then $G_1 \vee G_2$ is called the wheel, W_n , where $n > 2$.

Here we have proved that $C_n \vee W_m$ for $n \geq 3, m \geq 4$ and $K_n \vee W_m$ for $n \geq 2, m \geq 4$ are not HMC.

2. Main Results

Remark: Note that the wheel $W_n = C_n \vee K_1$. If all the vertices with label 2 of wheel W_n which are part of a cycle C_n , are in sequence in C_n , then it produces maximum number of edges with label 2 in wheel W_n and if all the vertices with label 1 of W_n which are part of a cycle C_n , are in sequence in C_n , then it produces minimum number of edges with label 1 in wheel W_n . So, this is the best possible way to minimize $|e_f(1) - e_f(2)|$ in the wheel graph W_n . So, through out this article, we have considered all the vertices of W_n with label 1 and label 2 which are part of cycle C_n are in sequence in C_n .

Theorem 2.1 $C_n \vee W_{n-1}$ is not HMC, where $n \geq 4$.

Proof: Suppose that $C_n \vee W_{n-1}$ is HMC. Note that, $|V(C_n \vee W_{n-1})| = 2n$ and $|E(C_n \vee W_{n-1})| = n + 2(n-1) + n^2$. Since, $|V(C_n \vee W_{n-1})| = 2n$ and we have assumed that $C_n \vee W_{n-1}$ is HMC. We have $v_f(1) = v_f(2) = n$.

Case 1: All the vertices in C_n have label 1 and all the vertices in W_{n-1} have label 2

Note that, $e_f(1) = n + n^2$ and $e_f(2) = 2(n-1)$. Now, $e_f(1) - e_f(2) = n^2 - n + 2 > 1$ as $n^2 \geq n$.

Case 2: All the vertices in C_n have label 2 and all the vertices in W_{n-1} have label 1

Then, we have $e_f(1) = 2(n-1) + n^2$ and $e_f(2) = n$.

Now, $e_f(1) - e_f(2) = n^2 + n - 2 > 1$ as $n \geq 4$

Case 3: We have t number of vertices with label 1 in C_n .

In this Case we have the following two possibilities.

(i) Apex vertex in W_{n-1} has label 1

(ii) Apex vertex in W_{n-1} has label 2

Subcase 3.1: Apex vertex in W_{n-1} has label 1

Suppose that we have t number of vertices with label 1 in C_n . So, we have $(n-t)$ number of vertices with label 1 in W_{n-1} . Hence, we have $(n-t)$ vertices with label 2 in C_n and t vertices with label 2 in W_{n-1} . Note that, $e_f(1) = (t+1) + tn + (n-1) + (n-t) + (n-t)^2$ and $e_f(2) = (n-t-1) + t(n-t) + (t-1)$. Then, $e_f(1) - e_f(2) = (n-t)^2 + n + t^2 + 2 > 1$ as $n \geq t$.

Subcase 3.2: Apex vertex in W_{n-1} has label 2

Suppose that we have t number of vertices with label 1 in C_n . So, we have $(n-t)$ number of vertices with label 1 in W_{n-1} . Hence, we have $(n-t)$ vertices with label 2 in C_n and t vertices with label 2 in W_{n-1} . Note that, $e_f(1) = (t+1) + tn + (n-t)^2 + 2(n-t) + 1$ and $e_f(2) = (n-t-1) + t(n-t) + (2t-3)$. Then, $e_f(1) - e_f(2) = (n-t)^2 + (t-1)^2 + n + 5 > 1$ as $n \geq t$ and $t \geq 1$. Hence, $C_n \vee W_{n-1}$ is not HMC, where $n \geq 4$.

Theorem 2.2 $C_n \vee W_m$ is not HMC, where $(m+n) \equiv 0 \pmod{2}, n < m-1, m \geq 4$ and $n \geq 3$.

Proof: Note that, $|V(C_n \vee W_m)| = m+n+1$ and $|E(C_n \vee W_m)| = n+n(m+1)+2m$. Suppose that $C_n \vee W_m$ is HMC. So, we have the following two possibilities.

(i) $|v_f(1)| = (m+n+2)/2$ and $|v_f(2)| = (m+n)/2$

(ii) $|v_f(1)| = (m+n)/2$ and $|v_f(2)| = (m+n+2)/2$

Case 1: $|v_f(1)| = (m+n+2)/2$ and $|v_f(2)| = (m+n)/2$

Subcase 1.1: Apex vertex in W_m has label 1

Subsubcase 1.1.1: All the vertices in C_n have label 1

We have n number of vertices with label 1 in C_n . So, we have $(m+n+2)/2 - n = (m-n+2)/2$ number of vertices with label 1 in W_m . Hence, we have $(m+n)/2$ number of vertices with label 2 in W_m . Then we have, $e_f(1) = n + (m + (m-n+2)/2) + n(m+1)$ and $e_f(2) = (m+n)/2 - 1$. Then $e_f(1) - e_f(2) = n + m + mn + 2 > 1$.

Subsubcase 1.1.2: All the vertices in C_n have label 2

We have n number of vertices with label 2 in C_n . So, we have $(m+n)/2 - n = (m-n)/2$ number of vertices with label 2 in W_m . Hence, we have $(m+n+2)/2$ number of vertices with label 1 in W_m . Note that, $e_f(1) = m + (m+n+2)/2 + n(m+n+2)/2$ and $e_f(2) = n + ((m-n)/2 - 1) + n(m-n)/2$. Then $e_f(1) - e_f(2) = n^2 + n + m + 2 > 1$.

Subsubcase 1.1.3: We have t number of vertices with label 1 in C_n .

So, we have $(m+n+2)/2 - t$ vertices with label 1 in W_m . Hence, we have $(n-t)$ vertices with label 2 in C_n and $m+1 - (m+n+2)/2 + t = (m-n)/2 + t$ vertices with label 2 in W_m . Note that $e_f(1) = (t+1) + t(m+1) + [m + (m+n+2)/2 - t] + (n-t)((m+n+2)/2 - t)$ and $e_f(2) = (n-t-1) + (n-t)((m-n)/2 + t) + (m-n)/2 + t - 1$.

Then, $e_f(1) - e_f(2) = (m-n)t + (n-t)^2 + m + n + t^2 + 4 > 1$ as $n \geq t$ and $n < m - 1$.

Subcase 1.2: Apex vertex in W_m has label 2

Subsubcase 1.2.1: All the vertices in C_n have label 1

We have n number of vertices with label 1 in C_n . So, we have $(m+n+2)/2 - n = (m-n+2)/2$ number of vertices with label 1 in W_m . Hence, we have $(m+n)/2$ number of vertices with label 2 in W_m . Then we have, $e_f(1) = n + [2((m-n+2)/2) + 1] + n(m+1)$ and $e_f(2) = m + n - 3$. Then, $e_f(1) - e_f(2) = mn + 6 > 1$.

Subsubcase 1.2.2: All the vertices in C_n have label 2

We have n number of vertices with label 2 in C_n . So, we have $(m+n)/2 - n = (m-n)/2$ number of vertices with label 2 in W_m . Hence, we have $(m+n+2)/2$ number of vertices with label 1 in W_m . Note that, $e_f(1) = n(n+m+2)/2 + [2((n+m+2)/2) + 1]$ and $e_f(2) = m - n - 3 + n(m-n)/2 + n$. Then, $e_f(1) - e_f(2) = n^2 + 2n + 6 > 1$.

Subsubcase 1.2.3: We have t number of vertices with label 1 in C_n .

So, we have $(m+n+2)/2 - t$ vertices with label 1 in W_m . Hence, we have $(n-t)$ vertices with label 2 in C_n and $m+1 - (m+n+2)/2 + t = (m-n)/2 + t$ vertices with label 2 in W_m . Note that $e_f(1) = (t+1) + (n-t)((m+n+2)/2 - t) + t(m+1) + 2((m+n+2)/2 - t) + 1$ and $e_f(2) = (n-t-1) + (n-t)((m-n)/2 + t) + m - n + 2t - 3$. Then, $e_f(1) - e_f(2) = 8 + 2(n-t) + 2n + (n-t)^2 + t^2 + (m-n)t > 1$ as $n \geq t$ and $n < m - 1$.

Case 2: $|v_f(1)| = (m+n)/2$ and $|v_f(2)| = (m+n+2)/2$

Subcase 2.1: Apex vertex in W_m has label 1

Subsubcase 2.1.1: All the vertices in C_n have label 1

We have n number of vertices with label 1 in C_n . So, we have $(m+n)/2 - n = (m-n)/2$ number of vertices with label 1 in W_m . Hence, we have $(m+n+2)/2$ number of vertices with label 2 in W_m . Note that, $e_f(1) = n + (m + (m-n)/2) + n(m+1)$ and $e_f(2) = ((m+n+2))/2 - 1$. Then, $e_f(1) - e_f(2) = n + m + nm > 1$.

Subsubcase 2.1.2: All the vertices in C_n have label 2

We have n number of vertices with label 2 in C_n . So, we have $(m+n+2)/2 - n = (m-n+2)/2$ number of vertices with label 2 in W_m . Hence, we have $(m+n)/2$ number of vertices with label 1 in W_m . Note that, $e_f(1) = m + (m+n)/2 + n((m+n)/2)$ and $e_f(2) = n + n(m-n+2)/2 + ((m-n+2)/2 - 1)$. Then, $e_f(1) - e_f(2) = (n^2 - n) + m > 1$.

Subsubcase 2.1.3: We have t number of vertices with label 1 in C_n .

So, we have $(m+n)/2 - t$ vertices with label 1 in W_m . Hence, we have $(n-t)$ vertices with label 2 in C_n and $m+1 - (m+n)/2 + t = (m-n)/2 + t + 1$ vertices with label 2 in W_m . Note that $e_f(1) = (t+1) + (n-t)((m+n)/2 - t) + t(m+1) + m + (m+n)/2 - t$ and $e_f(2) = (n-t-1) + (n-t)((m-n)/2 +$

$t + 1) + (m - n)/2 + t$. Then, $e_f(1) - e_f(2) = (n - t)^2 + (m - n)t + (m - n) + t^2 + 2t + 2 > 1$ as $n \geq t$ and $n < m - 1$.

Subcase 2.2: Apex vertex in W_m has label 2

Subsubcase 2.2.1: All the vertices in C_n have label 1

We have n number of vertices with label 1 in C_n . So, we have $(m + n)/2 - n = (m - n)/2$ number of vertices with label 1 in W_m . Hence, we have $(m + n + 2)/2$ number of vertices with label 2 in W_m . Note that, $e_f(1) = n + 2((m - n)/2) + 1 + n(m + 1)$ and $e_f(2) = m + n - 1$. Then, $e_f(1) - e_f(2) = mn + 2 > 1$.

Subsubcase 2.2.2: All the vertices in C_n have label 2

We have n number of vertices with label 2 in C_n . So, we have $(m + n + 2)/2 - n = (m - n + 2)/2$ number of vertices with label 2 in W_m . Hence, we have $(m + n)/2$ number of vertices with label 1 in W_m . Note that, $e_f(1) = n((m + n)/2) + 2((m + n)/2) + 1$ and $e_f(2) = n + n((m - n + 2)/2) + m - n - 1$. Then, $e_f(1) - e_f(2) = n^2 + 2 > 1$.

Subsubcase 2.2.3: We have t number of vertices with label 1 in C_n .

So, we have $(m + n)/2 - t$ vertices with label 1 in W_m . Hence, we have $(n - t)$ vertices with label 2 in C_n and $m + 1 - ((m + n)/2) + t = (m - n)/2 + t + 1$ vertices with label 2 in W_m . Note that $e_f(1) = (t + 1) + (n - t)((m + n)/2 - t) + t(m + 1) + 2((m + n)/2 - t) + 1$ and $e_f(2) = (n - t - 1) + (n - t)((m - n)/2 + t + 1) + m - n + 2t - 1$. Then, $e_f(1) - e_f(2) = (m - n)t + (n - t)^2 + t^2 + 4 > 1$ as $n \geq t$. Hence, $C_n \vee W_m$ is not HMC, where $(m + n) \equiv 0 \pmod{2}$, $n < m - 1$, $m \geq 4$ and $n \geq 3$.

Theorem 2.3 $C_n \vee W_m$ is not HMC, where $(m + n) \equiv 0 \pmod{2}$, $n > m - 1$, $m \geq 4$ and $n \geq 3$.

Proof: Note that, $|V(C_n \vee W_m)| = m + n + 1$ and $|E(C_n \vee W_m)| = n + n(m + 1) + 2m$. Suppose that $C_n \vee W_m$ is HMC. So, we have the following two possibilities.

(i) $|v_f(1)| = (m + n + 2)/2$ and $|v_f(2)| = (m + n)/2$

(ii) $|v_f(1)| = (m + n)/2$ and $|v_f(2)| = (m + n + 2)/2$

Case 1: $|v_{f(1)}| = (m + n + 2)/2$ and $|v_f(2)| = (m + n)/2$

Subcase 1.1: Apex vertex in W_m has label 1

Subsubcase 1.1.1: All the vertices in W_m have label 1

We have $(m + 1)$ number of vertices with label 1 in W_m . So, we have $(m + n + 2)/2 - (m + 1) = (n - m)/2$ number of vertices with label 1 in C_n . Hence, we have $(m + n)/2$ number of vertices with label 2 in C_n . Note that, $e_f(1) = ((n - m)/2 + 1) + n(m + 1) + 2m$ and $e_f(2) = (m + n)/2 - 1$. Then, $e_f(1) - e_f(2) = m + mn + n + 2 > 1$.

Subsubcase 1.1.2: All the vertices in W_m have label 2 except apex vertex

We have m number of vertices with label 2 in W_m . So, we have $(m + n)/2 - m = (n - m)/2$ number of vertices with label 2 in C_n . Hence, we have $(m + n)/2$ number of vertices with label 1 in C_n as apex vertex has label 1. Note that, $e_f(1) = (m + n)/2 + 1 + (m + 1)((m + n)/2) + (n - m)/2 + m$ and $e_f(2) = ((n - m)/2 - 1) + m((n - m)/2) + m$. Then, $e_f(1) - e_f(2) = m^2 + m + n + 2 > 1$.

Subsubcase 1.1.3: We have t number of vertices with label 1 in W_m .

So, we have $(m + n + 2)/2 - t$ vertices with label 1 in C_n . Hence, we have $(m + 1 - t)$ vertices with label 2 in W_m and $n - (m + n + 2)/2 + t = (n - m - 2)/2 + t$ vertices with label 2 in C_n . Note that $e_f(1) = (m + n + 2)/2 - t + 1 + m + t + tn + (m + 1 - t)((m + n + 2)/2 - t)$ and $e_f(2) = ((n - m - 2)/2 + t - 1) + m - t + (m + 1 - t)((n - m - 2)/2 + t)$. Then, $e_f(1) - e_f(2) = (m - t)^2 + t^2 + (n - m)t + 4(m - t) + 6 > 1$ as $n > m - 1$ and $m \geq t$.

Subcase 1.2: Apex vertex in W_m has label 2

Subsubcase 1.2.1: All the vertices in W_m have label 1 except apex vertex

We have m number of vertices with label 1 in W_m . So, we have $(m + n + 2)/2 - m = (n - m + 2)/2$ number of vertices with label 1 in C_n . Hence, we have $((m + n)/2 - 1)$ number of vertices with label 2 in C_n . Note that, $e_f(1) = 2m + ((n - m + 2)/2 + 1) + (m + 1)(n - m + 2)/2 + m((m + n)/2 -$

1) and $e_f(2) = (m+n)/2 - 3 + (m+n)/2$.

Then, $e_f(1) - e_f(2) = mn + 6 > 1$.

Subsubcase 1.2.2: All the vertices in W_m have label 2

We have $(m+1)$ number of vertices with label 2 in W_m . So, we have $(m+n)/2 - (m+1) = (n-m)/2 - 1$ number of vertices with label 2 in C_n . Hence, we have $((m+n+2)/2)$ number of vertices with label 1 in C_n .

Note that, $e_f(1) = ((m+n+2)/2 + 1) + (m+1)((m+n+2)/2)$ and $e_f(2) = (n-m)/2 - 2 + 2m + (m+1)((n-m)/2 - 1)$. Then, $e_f(1) - e_f(2) = 2m + m^2 + 6 > 1$.

Subsubcase 1.2.3: We have t number of vertices with label 1 in W_m .

So, we have $(m+n+2)/2 - t$ vertices with label 1 in C_n . Hence, we have $(m+1-t)$ vertices with label 2 in W_m and $n - (m+n+2)/2 + t = (n-m-2)/2 + t$ vertices with label 2 in C_n . Note that $e_f(1) = (m+n+2)/2 - t + 1 + 2t + 1 + tn + (m+1-t)((m+n+2)/2 - t)$ and $e_f(2) = ((n-m-2)/2 + t - 1) + 2m - 2t - 1 + (m+1-t)((n-m-2)/2 + t)$. Then, $e_f(1) - e_f(2) = (m-t)^2 + t^2 + (n-m)t + 2(m-t) + 8 > 1$ as $n > m - 1$ and $m \geq t$.

Case 2: $|v_f(1)| = (m+n)/2$ and $|v_f(2)| = (m+n+2)/2$

Subcase 2.1: Apex vertex in W_m has label 1

Subsubcase 2.1.1: All the vertices in W_m have label 1

We have $(m+1)$ number of vertices with label 1 in W_m . So, we have $(m+n)/2 - (m+1) = (n-m)/2 - 1$ number of vertices with label 1 in C_n . Hence, we have $((m+n+2)/2)$ number of vertices with label 2 in C_n . Note that, $e_f(1) = (n-m)/2 + n(m+1) + 2m$ and $e_f(2) = (m+n+2)/2 - 1$. Then, $e_f(1) - e_f(2) = mn + n + m > 1$.

Subsubcase 2.1.2: All the vertices in W_m have label 2 except apex vertex

We have m number of vertices with label 2 in W_m . Suppose that we have $(m+n+2)/2 - m = (n-m+2)/2$ number of vertices with label 2 in C_n . Hence, we have $((m+n)/2 - 1)$ number of vertices with label 1 in C_n . Note that, $e_f(1) = (m+n)/2 + m + ((m+n)/2 - 1)(m+1) + (n-m+2)/2$ and $e_f(2) = m + m((n-m+2)/2) + ((n-m+2)/2 - 1)$. Then, $e_f(1) - e_f(2) = m^2 + (n-m) > 1$ as $n > m - 1$.

Subsubcase 2.1.3: We have t number of vertices with label 1 in W_m .

So, we have $(m+n)/2 - t$ vertices with label 1 in C_n . Hence, we have $(m+1-t)$ vertices with label 2 in W_m and $n - ((m+n)/2) + t = (n-m)/2 + t$ vertices with label 2 in C_n . Note that $e_f(1) = ((m+n)/2 - t + 1) + m + t + tn + (m+1-t)((m+n)/2 - t)$ and $e_f(2) = ((n-m)/2 + t - 1) + (m-t) + (m+1-t)((n-m)/2 + t)$. Then, $e_f(1) - e_f(2) = (m-t)^2 + (n-m)t + t^2 + 2(m-t) + 2 > 1$ as $n > m - 1$ and $m \geq t$.

Subcase 2.2: Apex vertex in W_m has label 2

Subsubcase 2.2.1: All the vertices in W_m have label 1 except apex vertex

We have m number of vertices with label 1 in W_m . So, we have $(m+n)/2 - m = (n-m)/2$ number of vertices with label 1 in C_n . Hence, we have $((m+n)/2)$ number of vertices with label 2 in C_n . Note that, $e_f(1) = 2m + ((n-m)/2)(m+1) + m((m+n)/2) + (n-m)/2 + 1$ and $e_f(2) = (m+n)/2 - 1 + (m+n)/2$. Then, $e_f(1) - e_f(2) = mn + 2 > 1$.

Subsubcase 2.2.2: All the vertices in W_m have label 2

We have $(m+1)$ number of vertices with label 2 in W_m . So, we have $(m+n+2)/2 - (m+1) = (n-m)/2$ number of vertices with label 2 in C_n . Hence, we have $((m+n)/2)$ number of vertices with label 1 in C_n . Note that, $e_f(1) = (m+n)/2 + 1 + (m+1)((m+n)/2)$ and $e_f(2) = (n-m)/2 - 1 + (m+1)((n-m)/2) + 2m$. Then, $e_f(1) - e_f(2) = m^2 + 2 > 1$.

Subsubcase 2.2.3: We have t number of vertices with label 1 in W_m .

So, we have $(m+n)/2 - t$ vertices with label 1 in C_n . Hence, we have $(m+1-t)$ vertices with label 2 in W_m and $n - ((m+n)/2) + t = (n-m)/2 + t$ vertices with label 2 in C_n . Note that $e_f(1) = ((m+n)/2 - t + 1) + 2t + 1 + tn + (m+1-t)((m+n)/2 - t)$ and $e_f(2) = ((n-m)/2 + t - 1) +$

$(m - t) + (m - t - 1) + (m + 1 - t)((n - m)/2 + t)$. Then, $e_f(1) - e_f(2) = (m - t)^2 + (n - m)t + t^2 + 4 > 1$ as $n > m - 1$ and $m \geq t$. Hence, $C_n \vee W_m$ is not HMC, where $(m + n) \equiv 0 \pmod{2}$, $n > m - 1$, $m \geq 4$ and $n \geq 3$.

Theorem 2.4 $C_n \vee W_m$ is not HMC, where $(m + n) \equiv 1 \pmod{2}$, $n < m - 1$, $m \geq 4$ and $n \geq 3$.

Proof: Note that, $|V(C_n \vee W_m)| = m + n + 1$ and $|E(C_n \vee W_m)| = n + n(m + 1) + 2m$. Suppose that $C_n \vee W_m$ is HMC. Then we have $|v_f(1)| = (m + n + 1)/2 = |v_f(2)|$.

Case 1: Apex vertex in W_m has label 1

Subcase 1.1: All the vertices in C_n have label 1

We have n number of vertices with label 1 in C_n . So, we have $(m + n + 1)/2 - n = (m - n + 1)/2$ number of vertices with label 1 in W_m . Hence, we have $(m + n + 1)/2$ number of vertices with label 2 in W_m . Note that, $e_f(1) = n + n(m + 1) + m + (m - n + 1)/2$ and $e_f(2) = (m + n + 1)/2 - 1 = (m + n - 1)/2$. Then, $e_f(1) - e_f(2) = nm + n + m + 1 > 1$.

Subcase 1.2: All the vertices in C_n have label 2

We have n number of vertices with label 2 in C_n . So, we have $(m + n + 1)/2 - n = (m - n + 1)/2$ number of vertices with label 2 in W_m . Hence, we have $(m + n + 1)/2$ number of vertices with label 1 in W_m . Note that, $e_f(1) = n((m + n + 1)/2) + m + (m + n + 1)/2$ and $e_f(2) = n + (m - n + 1)/2 - 1 + n((m - n + 1)/2)$. Then, $e_f(1) - e_f(2) = m + n^2 + 1 > 1$.

Subcase 1.3: We have t number of vertices with label 1 in C_n .

So, we have $(m + n + 1)/2 - t$ vertices with label 1 in W_m . Hence, we have $(n - t)$ vertices with label 2 in C_n and $m + 1 - ((m + n + 1)/2) + t = (m - n + 1)/2 + t$ vertices with label 2 in W_m .

Note that $e_f(1) = (t + 1) + (n - t)((m + n + 1)/2 - t) + t(m + 1) + (m + n + 1)/2 - t + m$ and $e_f(2) = (n - t - 1) + (n - t)((m - n + 1)/2 + t) + (m - n + 1)/2 + t - 1$. Then, $e_f(1) - e_f(2) = (n - t)^2 + (m - n)t + m + t + t^2 + 3 > 1$ as $n \geq t$ and $n < m - 1$.

Case 2: Apex vertex in W_m has label 2

Subcase 2.1: All the vertices in C_n have label 1

We have n number of vertices with label 1 in C_n . So, we have $(m + n + 1)/2 - n = (m - n + 1)/2$ number of vertices with label 1 in W_m . Hence, we have $(m + n + 1)/2$ number of vertices with label 2 in W_m . Note that, $e_f(1) = n + n(m + 1) + 2((m - n + 1)/2) + 1$ and $e_f(2) = m + n - 2$. Then $e_f(1) - e_f(2) = mn + 4 > 1$.

Subcase 2.2: All the vertices in C_n have label 2

We have n number of vertices with label 2 in C_n . So, we have $((m + n + 1)/2 - n = (m - n + 1)/2$ number of vertices with label 2 in W_m . Hence, we have $(m + n + 1)/2$ number of vertices with label 1 in W_m . Note that, $e_f(1) = 2((m + n + 1)/2) + 1 + n((m + n + 1)/2)$ and $e_f(2) = n + n((m - n + 1)/2) + m - n - 2$. Then, $e_f(1) - e_f(2) = n^2 + n + 4 > 1$.

Subcase 2.3: We have t number of vertices with label 1 in C_n .

So, we have $(m + n + 1)/2 - t$ vertices with label 1 in W_m . Hence, we have $(n - t)$ vertices with label 2 in C_n and $m + 1 - ((m + n + 1)/2) + t = (m - n + 1)/2 + t$ vertices with label 2 in W_m .

Note that $e_f(1) = (t + 1) + (n - t)((m + n + 1)/2 - t) + t(m + 1) + 2((m + n + 1)/2 - t) + 1$ and $e_f(2) = (n - t - 1) + (n - t)((m - n + 1)/2 + t) + m - n - 2 + 2t$.

Then, $e_f(1) - e_f(2) = (n - t)^2 + (m - n)t + (t^2 - t) + n + 6 > 1$ as $n \geq t$ and $n < m - 1$.

Hence, $C_n \vee W_m$ is not HMC, where $(m + n) \equiv 1 \pmod{2}$, $n < m - 1$, $m \geq 4$ and $n \geq 3$.

Theorem 2.5 $C_n \vee W_m$ is not HMC, where $(m + n) \equiv 1 \pmod{2}$, $n > m - 1$, $m \geq 4$ and $n \geq 3$.

Proof: Note that, $|V(C_n \vee W_m)| = m + n + 1$ and $|E(C_n \vee W_m)| = n + n(m + 1) + 2m$. Suppose that $C_n \vee W_m$ is HMC. Then we have $|v_f(1)| = (m + n + 1)/2 = |v_f(2)|$.

Case 1: Apex vertex in W_m has label 1

Subcase 1.1: All the vertices in W_m have label 1

We have $(m + 1)$ number of vertices with label 1 in W_m . So, we have $(n - m - 1)/2$ number of vertices

with label 1 in C_n . Hence, we have $(m + n + 1)/2$ number of vertices with label 2 in C_n . Note that, $e_f(1) = ((n - m - 1)/2 + 1) + n(m + 1) + 2m$ and $e_f(2) = (n + m - 1)/2$. Then, $e_f(1) - e_f(2) = nm + m + n + 1 > 1$.

Subcase 1.2: All the vertices in W_m have label 2 except apex vertex

We have m number of vertices with label 2 in W_m . So, we have $(m + n + 1)/2 - m = (n - m + 1)/2$ number of vertices with label 2 in C_n . Hence, we have $(m + n - 1)/2$ number of vertices with label 1 in C_n .

Note that, $e_f(1) = (m + n - 1)/2 + 1 + (m + 1)((m + n - 1)/2) + (n - m + 1)/2 + m$

and $e_f(2) = (n - m + 1)/2 - 1 + m((n - m + 1)/2) + m$. Then, $e_f(1) - e_f(2) = m^2 + n + 1 > 1$.

Subcase 1.3: We have t number of vertices with label 1 in W_m .

So, we have $(m + n + 1)/2 - t$ vertices with label 1 in C_n . Hence, we have $(m + 1 - t)$ vertices with label 2 in W_m and $n - ((m + n + 1)/2) + t = (n - m - 1)/2 + t$ vertices with label 2 in C_n . Note that $e_f(1) = (m + n + 1)/2 - t + 1 + m + t + tn + (m + 1 - t)((m + n + 1)/2 - t)$ and $e_f(2) = (n - m - 1)/2 + t - 1 + (m - t) + (m + 1 - t)((n - m - 1)/2 + t)$. Then, $e_f(1) - e_f(2) = (m - t)^2 + 3(m - t) + t^2 + (n - m)t + 4 > 1$ as $m \geq t$.

Case 2: Apex vertex in W_m has label 2

Subcase 2.1: All the vertices in W_m have label 1 except apex vertex

We have m number of vertices with label 1 in W_m . So, we have $(m + n + 1)/2 - m = (n - m + 1)/2$ number of vertices with label 1 in C_n . Hence, we have $(m + n - 1)/2$ number of vertices with label 2 in C_n . Note that, $e_f(1) = ((n - m + 1)/2 + 1) + (m + 1)((n - m + 1)/2) + 2m + m((n + m - 1)/2)$ and $e_f(2) = (m + n - 1)/2 - 1 + (n + m - 1)/2$. Then, $e_f(1) - e_f(2) = mn + 4 > 1$.

Subcase 2.2: All the vertices in W_m have label 2

We have $(m + 1)$ number of vertices with label 2 in W_m . So, we have $(m + n + 1)/2 - (m + 1) = (n - m - 1)/2$ number of vertices with label 2 in C_n . So, we have $((m + n + 1)/2)$ number of vertices with label 1 in C_n . Note that, $e_f(1) = (m + n + 1)/2 + 1 + (m + 1)((m + n + 1)/2)$ and $e_f(2) = 2m + ((n - m - 1)/2)(m + 1) + (n - m - 1)/2 - 1$. Then, $e_f(1) - e_f(2) = m^2 + m + 4 > 1$.

Subcase 2.3: We have t number of vertices with label 1 in W_m .

So, we have $(m + n + 1)/2 - t$ vertices with label 1 in C_n . Hence, we have $(m + 1 - t)$ vertices with label 2 in W_m and $n - ((m + n + 1)/2) + t = (n - m - 1)/2 + t$ vertices with label 2 in C_n . Note that $e_f(1) = (m + n + 1)/2 - t + 1 + tn + 2t + 1 + (m + 1 - t)((m + n + 1)/2 - t)$ and $e_f(2) = (n - m - 1)/2 + t - 1 + (m - t) + (m - t - 1) + (m + 1 - t)((n - m - 1)/2 + t)$. Then, $e_f(1) - e_f(2) = (m - t)^2 + (n - m)t + m + t^2 - t + 4 > 1$ as $m \geq t$ and $n > m - 1$. Hence, $C_n \vee W_m$ is not HMC, where $(m + n) \equiv 1 \pmod{2}$, $n > m - 1$, $m \geq 4$ and $n \geq 3$.

Corollary 2.6 $C_n \vee W_m$ is not HMC, where ≥ 4 , $n \geq 3$.

Proof: Proof follows from Theorem 2.2, 2.3, 2.4, 2.5 and 2.6.

Theorem 2.7 $K_n \vee W_m$ is not HMC, where $(m + n) \equiv 0 \pmod{2}$ and $m \geq 4$, $n \geq 1$.

Proof: Note that, $|V(K_n \vee W_m)| = m + n + 1$ and $|E(K_n \vee W_m)| = n(n - 1)/2 + nm + 2m$. Suppose that $K_n \vee W_m$ is HMC. So, we have the following two possibilities,

(i) $|v_f(1)| = (m + n + 2)/2$ and $|v_f(2)| = (m + n)/2$

(ii) $|v_f(1)| = (m + n)/2$ and $|v_f(2)| = (m + n + 2)/2$

Case 1: $|v_f(1)| = (m + n + 2)/2$ and $|v_f(2)| = (m + n)/2$

Subcase 1.1: Apex vertex in W_m has label 1

Subsubcase 1.1.1: $n < m$ and all the vertices in K_n have label 1

Since we have n no. of vertices with label 1 in K_n , we have $(m + n + 2)/2 - n = (m - n + 2)/2$ no. of vertices with label 1 in W_m . Hence, we have $(m + n)/2$ no. of vertices with label 2 in W_m . Then we have, $e_f(1) = n(n - 1)/2 + m + (m - n + 2)/2 + n(m + 1)$ and $e_f(2) = (m + n)/2 - 1$. Now $e_f(1) - e_f(2) =$

$$(n^2 - n)/2 + m + mn + 2 > 1.$$

Subsubcase 1.1.2: $n < m$ and all the vertices in K_n have label 2

Since we have n no. of vertices with label 2 in K_n , we have $(m + n)/2 - n = (m - n)/2$ no. of vertices with label 2 in W_m . Hence, we have $(m + n + 2)/2$ no. of vertices with label 1 in W_m . Note that, $e_f(1) = m + (m + n + 2)/2 + n((m + n + 2)/2)$ and $e_f(2) = n(n - 1)/2 + (m - n)/2 - 1 + n(m - n)/2$. Now $e_f(1) - e_f(2) = m + n^2/2 + 5n/2 + 2 > 1$.

Subsubcase 1.1.3: $n > m$ and all the vertices in W_m have label 1

We have $(m + 1)$ no. of vertices with label 1 in W_m . So, we have $(m + n + 2)/2 - (m + 1) = (n - m)/2$ no. of vertices with label 1 in K_n . Hence, we have $(m + n)/2$ no. of vertices with label 2 in K_n .

Note that, $e_f(1) = (1/2)((n - m)/2)((n - m)/2 - 1) + ((n - m)/2)((m + n)/2) + n(m + 1) + 2m$ and $e_f(2) = (1/2)((m + n)/2)((m + n)/2 - 1)$. Then, $e_f(1) - e_f(2) = -m^2/4 + mn/2 + 5m/2 + n^2/4 + n = (n^2/2 - m^2/2) + mn/2 + 5m/2 + n > 1$ as $n > m - 1$.

Subsubcase 1.1.4: $n > m$ and all the vertices in W_m have label 2 except apex vertex

We have m no. of vertices with label 2 in W_m . So, we have $(m + n)/2 - m = (n - m)/2$ no. of vertices with label 2 in K_n . Hence, we have $(m + n + 2)/2 - 1 = (m + n)/2$ no. of vertices with label 1 in K_n . Note that, $e_f(1) = (1/2)((m + n)/2)((m + n)/2 - 1) + ((m + n)/2)(m + 1) + ((m + n)/2)((n - m)/2) + ((n - m)/2) + m$ and $e_f(2) = (1/2)((n - m)/2)$

$((n - m)/2 - 1) + m + m((n - m)/2)$. Then, $e_f(1) - e_f(2) = 3m^2/4 + mn/2 - m/2 + n^2/4 + n > 1$ as $mn > m$, $m \geq 4$ and $n \geq 1$.

Subsubcase 1.1.5: We have t no. of vertices with label 1 in K_n .

So, we have $(m + n + 2)/2 - t$ vertices with label 1 in W_m . Hence, we have $(n - t)$ vertices with label 2 in K_n and $m + 1 - ((m + n + 2)/2) + t = (m - n)/2 + t$ vertices with label 2 in W_m . Note that $e_f(1) = t(t - 1)/2 + t(n - t) + t(m + 1) + [m + ((m + n + 2)/2) - t] + (n - t)((m + n + 2)/2 - t)$ and $e_f(2) = (n - t)(n - t - 1)/2 + (n - t)((m - n)/2 + t) + (m - n)/2 + t - 1$. Now $e_f(1) - e_f(2) = (m - 3)t + m + 5n/2 + (1/2)[(n - t)^2 + t^2] + 2 > 1$ as $n \geq t$ and $m \geq 4$.

Subcase 1.2: Apex vertex in W_m has label 2

Subsubcase 1.2.1: $n < m$ and all the vertices in K_n have label 1

Since we have n no. of vertices with label 1 in K_n , we have $(m + n + 2)/2 - n = (m - n + 2)/2$ no. of vertices with label 1 in W_m . Hence, we have $(m + n)/2$ no. of vertices with label 2 in W_m . Note that $e_f(1) = n(n - 1)/2 + 2((m - n + 2)/2) + 1 + n(m + 1)$ and $e_f(2) = m + n - 3$. Now $e_f(1) - e_f(2) = (mn - m) + ((n^2/2) - n/2) + 6 > 1$ as $mn > m$ and $n^2 \geq n$.

Subsubcase 1.2.2: $n < m$ and all the vertices in K_n have label 2

Since we have n no. of vertices with label 2 in K_n , we have $(m + n)/2 - n = (m - n)/2$ no. of vertices with label 2 in W_m . Hence, we have $(m + n + 2)/2$ no. of vertices with label 1 in W_m . Note that, $e_f(1) = n((m + n + 2)/2) + n + m + 3$ and $e_f(2) = n(n - 1)/2 + m - n - 3 + n$. Now $e_f(1) - e_f(2) = n^2/2 + 7n/2 + 6 > 1$.

Subsubcase 1.2.3: $n > m$ and all the vertices in W_m have label 1 except apex vertex

Since m no. of vertices with label 1 in W_m . So, we have $(m + n + 2)/2 - m = (n - m + 2)/2$ no. of vertices with label 1 in K_n . Hence, we have $(m + n)/2 - 1 = (m + n - 2)/2$ no. of vertices with label 2 in K_n . Note that, $e_f(1) = (1/2)((n - m + 2)/2)((n - m + 2)/2 - 1) + ((n - m + 2)/2)((m + n - 2)/2) + ((n - m + 2)/2)(m + 1) + m(m + n - 2)/2 + 2m$ and $e_f(2) = (1/2)((m + n - 2)/2)((m + n - 2)/2 - 1) + (m + n - 2)/2$.

Then, $e_f(1) - e_f(2) = n^2/4 - m^2/4 + mn/2 + 5m/2 + n > 1$ as $n > m - 1$ and $n \geq 3$.

Subsubcase 1.2.4: $n > m$ and all the vertices in W_m have label 2

We have $(m + 1)$ no. of vertices with label 2 in W_m . So, we have $(m + n)/2 - (m + 1) = (n - m - 2)/2$ no. of vertices with label 2 in K_n . Hence, we have $(m + n + 2)/2$ no. of vertices with label 1 in K_n .

Note that, $e_f(1) = (1/2)((m + n + 2)/2)((m + n + 2)/2 - 1) + (m + 1)(m + n + 2)/2 + ((m + n + 2)/2)((n - m - 2)/2)$ and $e_f(2) = (1/2)((n - m - 2)/2)((n - m - 2)/2 - 1) + 2m + (m +$

1) $(n - m - 2)/2$.

Then, $e_f(1) - e_f(2) = 3m^2/4 - m/2 + mn/2 + n^2/4 + n > 1$ as $m^2 > m$.

Subsubcase 1.2.5: We have t no. of vertices with label 1 in K_n .

So, we have $(m + n + 2)/2 - t$ vertices with label 1 in W_m . Hence, we have $(n - t)$ vertices with label 2 in K_n and $m + 1 - ((m + n + 2)/2) + t = (m - n)/2 + t$ vertices with label 2 in W_m . Note that $e_f(1) = t(t - 1)/2 + t(n - t) + (n - t)((m + n + 2)/2 - t) + t(m + 1) + 2((m + n + 2)/2 - t) + 1$ and $e_f(2) = (n - t)(n - t - 1)/2 + (n - t)((m - n)/2 + t) + m - n + 2t - 3$. Then, $e_f(1) - e_f(2) = (mt - 4t) + (1/2)(n - t)^2 + t^2/2 + n/2 + (3n - t) + 6 > 1$ as $n \geq t$ and $m \geq 4$.

Case 2: $|v_f(1)| = (m + n)/2$ and $|v_f(2)| = (m + n + 2)/2$

Subcase 2.1: Apex vertex in W_m has label 1

Subsubcase 2.1.1: $n < m$ and all the vertices in K_n have label 1

We have n no. of vertices with label 1 in K_n . So, we have $(m + n)/2 - n = (m - n)/2$ no. of vertices with label 1 in W_m . Hence, we have $(m + n + 2)/2$ no. of vertices with label 2 in W_m . Note that, $e_f(1) = n(n - 1)/2 + m + (m - n)/2 + n(m + 1)$ and $e_f(2) = (m + n + 2)/2 - 1$. Then, $e_f(1) - e_f(2) = n(n - 1)/2 + m + nm > 1$.

Subsubcase 2.1.2: $n < m$ and all the vertices in K_n have label 2

We have n no. of vertices with label 2 in K_n . So, we have $(m + n + 2)/2 - n = (m - n + 2)/2$ no. of vertices with label 2 in W_m .

Hence, we have $(m + n)/2$ no. of vertices with label 1 in W_m . Note that, $e_f(1) = m + (m + n)/2 + n(m + n)/2$ and $e_f(2) = n(n - 1)/2 + n(m - n + 2)/2 + ((m - n + 2)/2 - 1)$. Then, $e_f(1) - e_f(2) = m + n^2/2 + n/2 > 1$.

Subsubcase 2.1.3: $n > m$ and all the vertices in W_m have label 1

We have $(m + 1)$ no. of vertices with label 1 in W_m . So, we have $(m + n)/2 - (m + 1) = (n - m - 2)/2$ no. of vertices with label 1 in K_n . Hence, we have $((m + n + 2)/2)$ no. of vertices with label 2 in K_n .

Then, $e_f(1) = (1/2)((n - m - 2)/2)((n - m - 2)/2 - 1) + ((n - m - 2)(m + n + 2)/4) + n(m + 1) + 2m$ and $e_f(2) = (1/2)((m + n + 2)/2)((m + n + 2)/2 - 1)$.

Then, $e_f(1) - e_f(2) = (n^2/4 - m^2/4) + mn/2 + 3m/2 > 1$ as $n > m - 1$.

Subsubcase 2.1.4: $n > m$ and all the vertices in W_m have label 2

We have m no. of vertices with label 2 in W_m . So, we have $((m + n + 2)/2 - m) = (n - m + 2)/2$ no. of vertices with label 2 in K_n .

Hence, we have $(m + n)/2 - 1 = (m + n - 2)/2$ no. of vertices with label 1 in K_n .

Then, $e_f(1) = (1/2)((m + n - 2)/2)((m + n - 2)/2 - 1) + ((m + n - 2)/2)((n - m + 2)/2) + (m + 1)((m + n - 2)/2) + ((n - m + 2)/2) + m$ and $e_f(2) = m + (1/2)((n - m + 2)/2)((n - m + 2)/2 - 1) + m(n - m + 2)/2$.

Then, $e_f(1) - e_f(2) = 3(m/2)^2 - 3m/2 + mn/2 + n^2/4 > 1$.

Subsubcase 2.1.5: We have t no. of vertices with label 1 in K_n .

So, we have $(m + n)/2 - t$ vertices with label 1 in W_m . Hence, we have $(n - t)$ vertices with label 2 in K_n and $m + 1 - ((m + n)/2) + t = (m - n)/2 + t + 1$ vertices with label 2 in W_m . Note that $e_f(1) = t(t - 1)/2 + t(n - t) + (n - t)((m + n)/2 - t) + t(m + 1) + m + (m + n)/2 - t$ and $e_f(2) = (n - t)(n - t - 1)/2 + (n - t)((m - n)/2 + t + 1) + (m - n)/2 + t$. Then, $e_f(1) - e_f(2) = (n - t)^2/2 + t^2/2 + (mt - t) + m + n/2 > 1$ as $n \geq t$ and $mt > t$.

Subcase 2.2: Apex vertex in W_m has label 2

Subsubcase 2.2.1: $n < m$ and all the vertices in K_n have label 1

We have n no. of vertices with label 1 in K_n . So, we have $(m + n)/2 - n = (m - n)/2$ no. of vertices with label 1 in W_m . Hence, we have $(m + n + 2)/2$ no. of vertices with label 2 in W_m . Note that, $e_f(1) = n(n - 1)/2 + 2((m - n)/2) + 1 + n(m + 1)$ and $e_f(2) = m + n - 1$. Then, $e_f(1) - e_f(2) = n(n - 1)/2 + 2 + (nm - n) > 1$ as $nm > n$.

Subsubcase 2.2.2: $n < m$ and all the vertices in K_n have label 2

We have n no. of vertices with label 2 in K_n . So, we have $(m+n+2)/2 - n = (m-n+2)/2$ no. of vertices with label 2 in W_m .

Hence, we have $(m+n)/2$ no. of vertices with label 1 in W_m . Note that, $e_f(1) = n((m+n)/2) + 2((m+n)/2) + 1$ and $e_f(2) = n(n-1)/2 + n((m-n+2)/2) + m - n - 1$. Then, $e_f(1) - e_f(2) = n^2/2 + 3n/2 + 2 > 1$.

Subsubcase 2.2.3: $n > m$ and all the vertices in W_m have label 1 except apex vertex

We have m no. of vertices with label 1 in W_m . So, we have $(m+n)/2 - m = (n-m)/2$ no. of vertices with label 1 in K_n .

Hence, we have $(m+n+2)/2 - 1 = (m+n)/2$ no. of vertices with label 2 in K_n . Note that, $e_f(1) = (1/2)((n-m)/2)((n-m)/2 - 1) + ((n-m)/2)((m+n)/2) + (m+1)(n-m)/2 + m(m+n)/2 + 2m$ and $e_f(2) = (1/2)((m+n)/2)((m+n)/2 - 1) + ((m+n)/2)$. Then, $e_f(1) - e_f(2) = n^2/4 - m^2/4 + mn/2 + 3m/2 > 1$ as $n > m$.

Subsubcase 2.2.4: $n > m$ and all the vertices in W_m have label 2

We have $(m+1)$ no. of vertices with label 2 in W_m . So, we have $(m+n+2)/2 - (m+1) = (n-m)/2$ no. of vertices with label 2 in K_n . Hence, we have $(m+n)/2$ no. of vertices with label 1 in K_n . Note that,

$e_f(1) = (1/2)((m+n)/2)((m+n)/2 - 1) + ((m+n)/2)(m+1) + ((m+n)/2)((n-m)/2)$ and $e_f(2) = (1/2)((n-m)/2)((n-m)/2 - 1) + ((n-m)/2)(m+1) + 2m$.

Then, $e_f(1) - e_f(2) = 3(m/2)^2 - 3m/2 + mn/2 + n^2/4 > 1$.

Subsubcase 2.2.5: We have t no. of vertices with label 1 in K_n .

So, we have $(m+n)/2 - t$ vertices with label 1 in W_m .

Hence, we have $(n-t)$ vertices with label 2 in K_n and $m+1 - ((m+n)/2) + t = (m-n)/2 + t + 1$ vertices with label 2 in W_m .

Note that $e_f(1) = t(t-1)/2 + t(n-t) + (n-t)((m+n)/2 - t) + t(m+1)$

$+ 2((m+n)/2 - t) + 1$ and $e_f(2) = (n-t)(n-t-1)/2 + (n-t)((m-n)/2 + t + 1) + m - n + 2t - 1$. Then, $e_f(1) - e_f(2) = (n-t)^2/2 + t^2/2 + 3n/2 + t(m-3) + 2 > 1$ as $m \geq 4$ and $n \geq t$.

Hence, $K_n \vee W_m$ is not HMC, where $(m+n) \equiv 0 \pmod{2}$ and $m \geq 4, n \geq 1$.

Theorem 2.8 $K_n \vee W_m$ is not HMC, where $(m+n) \equiv 1 \pmod{2}$ and $m \geq 4, n \geq 1$.

Proof: Note that, $|V(K_n \vee W_m)| = m+n+1$ and $|E(K_n \vee W_m)| = n(n-1)/2 + nm + 2m$. Suppose that $K_n \vee W_m$ is HMC. Then we have $|v_f(1)| = (m+n+1)/2 = |v_f(2)|$.

Case 1: Apex vertex in W_m has label 1

Subcase 1.1: $n < m$ and all the vertices in K_n have label 1

We have n no. of vertices with label 1 in K_n . So, we have $(m+n+1)/2 - n = (m-n+1)/2$ no. of vertices with label 1 in W_m . Hence, we have $(m+n+1)/2$ no. of vertices with label 2 in W_m . Note that,

$e_f(1) = n(n-1)/2 + n(m+1) + m + (m-n+1)/2$ and $e_f(2) = (m+n+1)/2 - 1 = (m+n-1)/2$.

Then, $e_f(1) - e_f(2) = (n^2 - n)/2 + nm + m + 1 > 1$ as $n^2 \geq n$.

Subcase 1.2: $n < m$ and all the vertices in K_n have label 2

We have n no. of vertices with label 2 in K_n . So, we have $(m+n+1)/2 - n = (m-n+1)/2$ no. of vertices with label 2 in W_m . Hence, we have $(m+n+1)/2$ no. of vertices with label 1 in W_m . Note that,

$e_f(1) = n(m+n+1)/2 + (m+n+1)/2 + m$ and $e_f(2) = n(n-1)/2 + n(m-n+1)/2 + (m-n+1)/2 - 1$. Then, $e_f(1) - e_f(2) = m + n^2/2 + 3n/2 + 1 > 1$.

Subcase 1.3: $n > m$ and all the vertices in W_m have label 1

We have $(m+1)$ no. of vertices with label 1 in W_m . Suppose that we have $(m+n+1)/2 - (m+1) = (n-m-1)/2$ no. of vertices with label 1 in K_n . Hence, we have $(m+n+1)/2$ no. of vertices with label 2 in K_n .

Note that, $e_f(1) = (1/2)((n-m-1)/2)((n-m-1)/2 - 1) + ((n-m-1)/2)$

$((m+n+1)/2) + n(m+1) + 2m$ and $e_f(2) = (1/2)((m+n+1)/2)((m+n+1)/2 - 1)$.

Then, $e_f(1) - e_f(2) = (n^2/4 - m^2/4) + mn/2 + n/2 + 2m + 1/4 > 1$ as $n > m$ and $n \geq 1$.

Subcase 1.4: $n > m$ and all the vertices in W_m have label 2 except apex vertex

We have m no. of vertices with label 2 in W_m .

So, we have $(m + n + 1)/2 - m = (n - m + 1)/2$ no. of vertices with label 2 in K_n .

Hence, we have $((m + n + 1)/2 - 1) = (m + n - 1)/2$ no. of vertices with label 1 in K_n .

Note that, $e_f(1) = (1/2)((m + n - 1)/2)((m + n - 1)/2 - 1) + ((m + n - 1)/2)$

$((n - m + 1)/2) + ((m + n - 1)/2)(m + 1) + (n - m + 1)/2 + m$ and $e_f(2) =$

$(1/2)((n - m + 1)/2)((n - m + 1)/2 - 1) + ((n - m + 1)/2)m + m$.

Then, $e_f(1) - e_f(2) = 3m^2/4 - m + n^2/4 + mn/2 + n/2 + 1/4 > 1$ as $m^2 > m$.

Subcase 1.5: We have t no. of vertices with label 1 in K_n .

So, we have $(m + n + 1)/2 - t$ vertices with label 1 in W_m . Hence, we have $(n - t)$ vertices with label 2

in K_n and $m + 1 - (m + n + 1)/2 + t = (m - n + 1)/2 + t$ vertices with label 2 in W_m . Note that $e_f(1) =$

$t(t - 1)/2 + t(n - t) + (n - t)((m + n + 1)/2 - t) + t(m + 1) + (m + n + 1)/2 - t + m$ and $e_f(2) =$

$(n - t)(n - t - 1)/2 + (n - t)((m - n + 1)/2 + t) + (m - n + 1)/2 + t - 1$. Then, $e_f(1) - e_f(2) =$

$(n - t)^2/2 + t^2/2 + (m - 2)t + m + 3n/2 + 1 > 1$ and $n \geq t$ and $m \geq 4$.

Case 2: Apex vertex in W_m has label 2

Subcase 2.1: $n < m$ and all the vertices in K_n have label 1

We have n no. of vertices with label 1 in K_n . So, we have $(m + n + 1)/2 - n = (m - n + 1)/2$ no. of

vertices with label 1 in W_m . Hence, we have $(m + n + 1)/2$ no. of vertices with label 2 in W_m . Note that,

$e_f(1) = n(n - 1)/2 + n(m + 1) + 2((m - n + 1)/2) + 1$ and $e_f(2) = m + n - 2$. Then $e_f(1) - e_f(2) =$

$n^2 - n/2 + (nm - n) + 4 > 1$ as $nm > n, n^2 \geq n$ and $n \geq 1$.

Subcase 2.2: $n < m$ and all the vertices in K_n have label 2

Suppose that K_n has all the vertices with label 2. So, we have $(m + n + 1)/2 - n = (m - n + 1)/2$ no. of

vertices with label 2 in W_m . Hence, we have $(m + n + 1)/2$ no. of vertices with label 1 in W_m . Note that,

$e_f(1) = 2((m + n + 1)/2) + 1 + n((m + n + 1)/2)$ and $e_f(2) = n(n - 1)/2 + n(m - n + 1)/2 + m -$

$n - 2$. Then, $e_f(1) - e_f(2) = n^2/2 + 5n/2 + 4 > 1$.

Subcase 2.3: $n > m$ and all the vertices in W_m have label 1 except apex vertex

We have m no. of vertices with label 1 in W_m . So, we have $(m + n + 1)/2 - m = (n - m + 1)/2$ no. of

vertices with label 1 in K_n . So, we have $(m + n + 1)/2 - 1 = (m + n - 1)/2$ no. of vertices with label 2

in K_n . Note that, $e_f(1) = (1/2)((n - m + 1)/2)((n - m + 1)/2 - 1) +$

$((n - m + 1)/2)((m + n - 1)/2) + ((n - m + 1)/2)(m + 1) + m(m + n - 1)/2 + 2m$ and $e_f(2) =$

$(1/2)((m + n - 1)/2)((m + n - 1)/2 - 1) + (m + n - 1)/2$.

Then, $e_f(1) - e_f(2) = (n^2/4 - m^2/4) + mn/2 + 2m + n/2 + 1/4 > 1$ as $n > m$.

Subcase 2.4: $n > m$ and all the vertices in W_m have label 2

We have $(m + 1)$ no. of vertices with label 2 in W_m . Suppose that we have $(m + n + 1)/2 - (m + 1) =$

$(n - m - 1)/2$ no. of vertices with label 2 in K_n . So, we have $(m + n + 1)/2$ no. of vertices with label 1

in K_n .

Note that, $e_f(1) = (1/2)((m + n + 1)/2)((m + n + 1)/2 - 1) + ((m + n + 1)/2)$

$((n - m - 1)/2) + ((m + n + 1)/2)(m + 1)$ and $e_f(2) = (1/2)((n - m - 1)/2)$

$((n - m - 1)/2 - 1) + (m + 1)(n - m - 1)/2 + 2m$. Then, $e_f(1) - e_f(2) = m^2/2 + (m^2 - m) + n^2/4 +$

$mn/2 + n/2 + 1/4 > 1$ as $m^2 > m$.

Subcase 2.5: We have t no. of vertices with label 1 in K_n .

So, we have $(m + n + 1)/2 - t$ vertices with label 1 in W_m . Hence, we have $(n - t)$ vertices with label 2

in K_n and $m + 1 - (m + n + 1)/2 + t = (m - n + 1)/2 + t$ vertices with label 2 in W_m . Note that $e_f(1) =$

$t(t - 1)/2 + t(n - t) + (n - t)((m + n + 1)/2 - t) + t(m + 1) + 2((m + n + 1)/2 - t) + 1$ and $e_f(2) =$

$(n - t)(n - t - 1)/2 + (n - t)((m - n + 1)/2 + t) + m - n + 2t - 2$. Then, $e_f(1) - e_f(2) = (m - 4)t +$

$(n - t)^2/2 + 5n/2 + t^2/2 + 4 > 1$ as $m \geq 4$ and $n \geq t$. Hence, $K_n \vee W_m$ is not HMC, where $(m + n) \equiv$

$1 \pmod{2}$ and $m \geq 4, n \geq 1$.

Corollary 2.9 $K_n \vee W_m$ is not HMC, where $m \geq 4$ and $n \geq 1$.

Proof: Proof follows from Propositions 2.8.

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